Exact stationary state for a deterministic high-speed traffic model with open boundaries

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Abstract
An exact solution for a high-speed deterministic traffic model with open
boundaries and synchronous update rule is presented. Due to the strong
correlations in the model, the qualitative structure of the stationary state can
be described for general values of the maximum speed. It is shown in the case
of $v_{max} = 2$ that a detailed analysis of this structure leads to an exact solution.
Explicit expressions for the stationary state probabilities are given in terms of
products of 24×24 matrices. From this solution an exact expression for the
correlation length is derived.

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1. Introduction

One-dimensional driven diffusive processes have proven to be an interesting playground for
the study of non-equilibrium behaviour [1–5]. Of great interest is the fact that the study of
many stationary state properties has come within reach of exact analytical methods since the
solution of the asymmetric simple exclusion process (ASEP) with random sequential update
and open boundaries [6–9]. An important analytical tool in the study of these diffusive systems
is the matrix product method, that appeared earlier in the study of lattice animals [10] and the
ground states of antiferromagnets [11, 12]. Its use in [9] for the ASEP has boosted a lot of
research on a variety of diffusion models, among which are, for example, the ASEP with
other updates [13–19], multi-species models [20–29], multi-lane traffic [30] and the partially
asymmetric exclusion process [31–34]. For recent reviews of many of the exact results for the
ASEP see [35–37].

It has been shown for different dynamical update rules, that the stationary state of a
stochastic model can always be written as a matrix product [38–40], although no proof is given
for the synchronous update. This mere fact by no means solves the problem of finding the
stationary state, but it provides a basis for a systematic study via the representation theory of
non-linear algebras [25, 26, 32]. In almost all the cases studied so far, the algebra has been
quadratic, which is peculiar to systems with nearest neighbour interactions only. Only for

synchronous ASEP, non-trivial representations of an algebra of higher degree have been used, i.e. either quartic [17], where the matrices depend on one site, or cubic [18] in the case where the matrices depend on two sites.

Recently the asymmetric exclusion process with next-nearest neighbour interactions has been studied by various methods [41], but in the case of open boundaries exact results have been obtained only on a special line. In an initial attempt to find exact stationary states for models with long range interactions and open boundaries, a deterministic high-speed asymmetric exclusion model is studied. Particles are allowed to hop over more than one lattice spacing per time step, and they enter the system at the left and leave at the right. Furthermore, the system will be subject to a synchronous dynamical update rule. For such dynamics the correlations are the strongest, which is not only interesting from a physical point of view, but actually helps solve the problem. The correlations are so strong that one can describe the stationary state qualitatively in a simple way by identifying so-called Garden of Eden states [42, 43].

The exact stationary state is given in matrix product form for the case where particles may hop over two lattice sites. The matrices depend on three sites and it is shown that they should satisfy an algebra at least quartic in degree. The model is solved by making an Ansatz for the form of the matrices based on the qualitative observations for the stationary state. It is then shown that the submatrices in this Ansatz must satisfy certain relations for the matrix product state so as to solve the stationary master equation. A solution of these relations is found with the help of explicit calculations for small systems.

This paper is organized as follows. The model is defined in detail in section 2. The exact stationary state is calculated in section 3 and some results on the phase diagram are discussed in section 4.

2. Definition of the model

In this paper we study a one-dimensional asymmetric particle hopping model where particles in the bulk hop to the right. Particles may enter the system at the left and leave at the right. In the bulk all particles will move with their maximum possible speed, which is either given by the speed limit $v_{\max}$, or is given by the distance to the next particle to avoid collisions. There will be no stochasticity in the bulk and particles always achieve their maximum possible speed instantaneously. In the case of periodic boundary conditions this model is known as the deterministic Fukui–Ishibashi model [44], for which some exact results are known [45].

With open boundaries, particles will be allowed to enter the system on the initial $v_{\max}$ sites and may leave the system from the final $v_{\max}$ sites. The choice of boundary conditions can have a profound influence on the behaviour of the system. If, for example, particles were allowed to enter only at the first site, the density profile for $v_{\max} > 1$ in the free flow phase would show a strong sublattice dependence. Moreover, in this case the system would not be able to reach its maximum possible flow, since particles would have to wait an extra time step due to the synchronous update before the first site is unblocked. In the case of $v_{\max} = 2$, the specific boundary conditions we will use here are similar to those of the random sequential model A of [40]. If the two sites at the left boundary are empty, a particle can enter on the second site with probability α_2, and on the first site with probability $\alpha_1(1 - \alpha_2)$. The sites remain empty with probability $(1 - \alpha_1)(1 - \alpha_2)$. If a particle is already present on the second site, a probability α_3 is given for a particle entering on the first site. At the right boundary, a particle at the last site will leave the system with probability β_1. If the last site is empty, but a particle is present on the penultimate site, it will leave the system with probability β_2. In terms of Boolean variables τ_i that have the value of 1 for a particle and 0 for a hole, the dynamical

rule for the bulk can be written as

$$\tau_i' = \tau_{i-2}\sigma_{i-1}\sigma_i + \tau_{i-1}\sigma_i\tau_{i+1} + \tau_i\tau_{i+1} \tag{1}$$

where the prime denotes time incremented by 1 and $\sigma = 1 - \tau$. At the boundaries the additional Boolean variables $\hat{\alpha}$ and $\hat{\beta}$ are used that have time averages equal to α and β. At the left boundary the rules are

$$\tau_1' = \tau_1\tau_2 + \hat{\alpha}_1(1 - \hat{\alpha}_2)\sigma_1\sigma_2 + \hat{\alpha}_3\sigma_1\tau_2 \tag{2}$$

$$\tau_2' = \tau_1\sigma_2\tau_3 + \tau_2\tau_3 + \hat{\alpha}_2\sigma_1\sigma_2 \tag{3}$$

and at the right boundary we have

$$\tau_L' = \tau_{L-2}\sigma_{L-1}\sigma_L + (1 - \hat{\beta}_1)\tau_L + (1 - \hat{\beta}_2)\tau_{L-1}\sigma_L. \tag{4}$$

The currents are defined by the continuity equation

$$\tau_i' = \tau_i + j_{i-1} - j_i \tag{5}$$

and are given by

$$\begin{aligned}
j_0 &= \hat{\alpha}_3\sigma_1\tau_2 + (1 - (1 - \hat{\alpha}_1)(1 - \hat{\alpha}_2))\sigma_1\sigma_2 \\
j_1 &= \tau_1\sigma_2 + \hat{\alpha}_2\sigma_1\sigma_2 \\
j_i &= \tau_i\sigma_{i+1} + \tau_{i-1}\sigma_i\sigma_{i+1} \\
j_L &= \hat{\beta}_1\tau_L + \hat{\beta}_2\tau_{L-1}\sigma_L.
\end{aligned} \tag{6}$$

For technical convenience we will put $\langle \hat{\beta}_1 \rangle = \langle \hat{\beta}_2 \rangle = \beta$, and $\langle \hat{\alpha}_1 \rangle = \langle \hat{\alpha}_2 \rangle = \langle \hat{\alpha}_3 \rangle = \alpha$ in the rest of this paper. All arguments, however, hold for the more general case. The calculation becomes more cumbersome and one has to discriminate between even and odd sublattices.

3. The stationary state

In the following discussion, the relative weight of a particular configuration $\{\tau_1, \ldots, \tau_L\}$ in the stationary state will be denoted by $P(\tau_1, \ldots, \tau_L)$. Once all relative weights are determined, the normalization Z_L can be calculated via

$$Z_L = \sum_{\{\tau\}} P(\tau_1, \ldots, \tau_L). \tag{7}$$

To derive some general conclusions about the stationary state, it is helpful to consider the extreme cases $\alpha = 1$ and $\beta = 1$ first.

3.1. Free flow

Since we are considering a deterministic model, spontaneous jams do not occur. Jams will only build up from obstacles at the right boundary. Pure free flow configurations are obtained by removing these obstacles, i.e. $\beta = 1$. The dynamical rule at the right boundary then becomes

$$\tau_L' = \tau_{L-2}\sigma_{L-1}\sigma_L. \tag{8}$$

Since there will be no jams in the stationary state, its bulk dynamics is given by

$$\tau_i' = \tau_{i-2} \tag{9}$$

and it follows that the master equation can be written as

$$P(\tau_1, \ldots, \tau_L) = \sum_{\mu, \mu'} p_f(\tau_1\tau_2\tau_3)p_f(\tau_2\tau_3\tau_4)P(\tau_3, \ldots, \tau_L, \mu, \mu') \tag{10}$$

where

$$p_f(000) = (1 - \alpha) \qquad p_f(100) = \alpha$$
$$p_f(010) = 1 \qquad p_f(001) = 1 \tag{11}$$
$$p_f(\tau\tau'\tau'') = 0 \quad \text{otherwise.}$$

This equation can be solved by the Ansatz

$$P(\tau_1, \ldots, \tau_L) = R(\tau_{L-1}\tau_L) \prod_{i=1}^{L-2} p_f(\tau_i \tau_{i+1} \tau_{i+2}) \tag{12}$$

and we find

$$R(00) = 1 \qquad R(10) = R(01) = \alpha. \tag{13}$$

3.2. Jammed flow

Pure jammed flow configurations are obtained by setting $\alpha = 1$. From dynamical rules it follows that in this case configurations with the sequence 000 in it do not occur in the stationary state. This means that the bulk and left boundary dynamics may be replaced by the simple rule

$$\tau_i' = \tau_{i+1}. \tag{14}$$

The master equation for this case is

$$P(\tau_1, \ldots, \tau_L) = \sum_\mu p_j(\tau_{L-2}\tau_{L-1}\tau_L) P(\mu, \tau_1, \ldots, \tau_{L-1}) \tag{15}$$

where

$$p_j(100) = \beta \qquad p_j(010) = \beta$$
$$p_j(001) = 1 \qquad p_j(110) = \beta$$
$$p_j(101) = 1 - \beta \qquad p_j(011) = 1 - \beta \tag{16}$$
$$p_j(111) = 1 - \beta \qquad p_j(\tau\tau'\tau'') = 0 \quad \text{otherwise.}$$

Again this equation can be solved by a simple Ansatz

$$P(\tau_1, \ldots, \tau_L) = L(\tau_1\tau_2) \prod_{i=1}^{L-2} p_j(\tau_i \tau_{i+1} \tau_{i+2}). \tag{17}$$

In this case we find

$$L(00) = \beta^2 \qquad L(10) = L(01) = \beta \qquad L(11) = 1 - \beta. \tag{18}$$

3.3. The general case

As mentioned already, spontaneous jams will not occur since we are considering a deterministic model. Following a similar line of reasoning as in [43], this can be deduced from the microscopic dynamics (1)–(4).

(i) The sequence 1100 can only arise from the same sequence shifted by one lattice unit

$$(\tau_i \tau_{i+1} \sigma_{i+2} \sigma_{i+3})' = \tau_i' \tau_{i+1} \tau_{i+2} \sigma_{i+3} \sigma_{i+4}. \tag{19}$$

Since

$$(\tau_{L-3}\tau_{L-2}\sigma_{L-1}\sigma_L)''' \sim (\tau_{L-2}\tau_{L-1}\sigma_L)'' \sim (\tau_{L-1}\tau_L)' = 0 \tag{20}$$

it follows that configurations with the sequence 1100 do not occur in the stationary state.

(ii) Similarly, a sequence 10 100 can only arise from the same sequence shifted by one lattice unit or from a sequence with 1100 in it

$$(\tau_i \sigma_{i+1} \tau_{i+2} \sigma_{i+3} \sigma_{i+4})' = (\tau_{i-2} \sigma_{i-1} + \tau_{i-1} \sigma_i)(\tau_{i+1} \sigma_{i+2} \tau_{i+3} \sigma_{i+4} \sigma_{i+5})$$
$$+ \tau_i' \sigma_{i+1} \tau_{i+2} \tau_{i+3} \sigma_{i+4} \sigma_{i+5}. \tag{21}$$

Since

$$(\tau_{L-4} \sigma_{L-3} \tau_{L-2} \sigma_{L-1} \sigma_L)''' \sim (\tau_{L-2} \sigma_{L-1} \tau_L)' = 0 \tag{22}$$

it follows from the previous observation that configurations also with the sequence 10 100 do not occur in the stationary state.

It thus follows that each configuration can be divided into three parts. The first part is a free flow part where there are *at least* two holes between successive particles. This part ends at site f which denotes the last site of the last 000 sequence of a configuration. The dynamics for this part is given by

$$\tau_i' = \tau_{i-2} \qquad 3 \leqslant i \leqslant f. \tag{23}$$

The third part starts at site j which denotes the first site of the first jammed configuration, i.e. a 11 or a 101 sequence, whichever comes first. This part is a jammed flow part where there are *at most* two holes between successive particles. For this part the dynamics is

$$\tau_i' = \tau_{i+1} \qquad j \leqslant i \leqslant L - 1. \tag{24}$$

In between these two parts there may be a configuration of 100 sequences of arbitrary length. A general configuration may thus be written as

$$\tau_1 \ldots \tau_f (100)^n \tau_j \ldots \tau_L. \tag{25}$$

This analysis can be performed in a similar way for models with higher v_{max} values. The free flow part will end with $v_{max} + 1$ zeros and the intermediate part will consist of a sequence of blocks, where each block starts with a 1 followed by v_{max} zeros. The jammed flow part starts with any of the local jammed configurations. These are those sequences where there are less than v_{max} zeros in between two 1s.

The master equation for the stationary state can be written explicitly in this notation. In the case where the jammed flow starts with a 11 pair it is given by

$$P(\tau_1 \ldots \tau_f (100)^n 11 \tau_{j+2} \ldots \tau_L) = p_f(\tau_1 \tau_2 \tau_3) p_f(\tau_2 \tau_3 \tau_4) p_j(\tau_{L-2} \tau_{L-1} \tau_L)$$

$$\times \Bigg[P(\tau_3 \ldots \tau_f (100)^{n+1} 11 \tau_{j+2} \ldots \tau_{L-1})$$

$$+ \sum_{p=0}^{n} P(\tau_3 \ldots \tau_f (100)^p 001 (100)^{n-p} 11 \tau_{j+2} \ldots \tau_{L-1})$$

$$+ \sum_{p=0}^{n} P(\tau_3 \ldots \tau_f (100)^p 010 (100)^{n-p} 11 \tau_{j+2} \ldots \tau_{L-1}) \Bigg]. \tag{26}$$

A slightly different equation is obtained when the jammed flow starts with a 101 sequence

$$P(\tau_1 \ldots \tau_f (100)^n 101 \tau_{j+3} \ldots \tau_L) = p_f(\tau_1 \tau_2 \tau_3) p_f(\tau_2 \tau_3 \tau_4) p_j(\tau_{L-2} \tau_{L-1} \tau_L)$$

$$\times \Bigg[P(\tau_3 \ldots \tau_f (100)^{n+1} 101 \tau_{j+3} \ldots \tau_{L-1})$$

$$+ \sum_{p=0}^{n} P(\tau_3 \ldots \tau_f (100)^p 001 (100)^{n-p} 101 \tau_{j+3} \ldots \tau_{L-1})$$

$$+ \sum_{p=0}^{n} P(\tau_3 \ldots \tau_f (100)^p 010(100)^{n-p} 101 \tau_{j+3} \ldots \tau_{L-1})$$

$$+ P(\tau_3 \ldots \tau_f (100)^{n+1} 011 \tau_{j+3} \ldots \tau_{L-1}) \Bigg]. \tag{27}$$

Similar equations are obtained when f and/or j are close to the boundary.

To solve (26) and (27) we will employ the powerful matrix product method [9]. The relative probabilities for the stationary are written as

$$P(\tau_1, \ldots, \tau_L) = \langle \mathcal{L}(\tau_1 \tau_2)| \prod_{i=1}^{L-2} \mathcal{M}(\tau_i \tau_{i+1} \tau_{i+2})|\mathcal{R}(\tau_{L-1} \tau_L)\rangle. \tag{28}$$

Due to the specific form of each configuration, the following triangular form for the matrices $\mathcal{M}$ suggests itself and is similar to the $v_{\max} = 1$ case [18]:

$$\mathcal{M}(\tau \tau' \tau'') = \begin{pmatrix} p_{\mathrm{f}}(\tau \tau' \tau'') F & \mathcal{S}(\tau \tau' \tau'') \\ 0 & p_{\mathrm{j}}(\tau \tau' \tau'') J \end{pmatrix} \tag{29}$$

where $p_{\mathrm{f}}(\tau \tau' \tau'')$ and $p_{\mathrm{j}}(\tau \tau' \tau'')$ are defined by (11) and (16), respectively. The matrices F and J are yet to be determined. While for $v_{\max} = 1$ they are just scalars given by $F = \beta$ and $J = \alpha$, they will be more complicated in the present case. The matrices $\mathcal{S}(\tau \tau' \tau'')$ will solve the dynamical equations for the bulk. They are defined on the interface only and $\mathcal{S}(000) = \mathcal{S}(110) = \mathcal{S}(101) = \mathcal{S}(011) = \mathcal{S}(111) = 0$. A similar decomposition as in (29) will be used for the boundary vectors

$$\langle \mathcal{L}(\tau_1 \tau_2)| = (\langle \mathcal{L}_{\mathrm{F}}(\tau_1 \tau_2)|, \langle \mathcal{L}_{\mathrm{J}}(\tau_1 \tau_2)|) \tag{30}$$

and likewise for $|\mathcal{R}(\tau_{L-1} \tau_L)\rangle$.

To make the following more transparent, we will use the notation $\mathcal{S}_1 = \mathcal{S}(100)$, $\mathcal{S}_2 = \mathcal{S}(010)$ and $\mathcal{S}_3 = \mathcal{S}(001)$. Upon substitution one quickly concludes that (26) and (27) are equivalent if

$$\mathcal{S}_2 J = \alpha F(\mathcal{S}_2 + (1-\alpha)\mathcal{S}_3). \tag{31}$$

Let us assume that this relation is satisfied and concentrate on (26). Substituting (29) in (26) one finds that

$$\begin{aligned} \langle \mathcal{L}_{\mathrm{F}}(00)| = \langle \mathcal{L}_{\mathrm{F}}(10)| = \langle \mathcal{L}_{\mathrm{F}}(01)| = \langle \mathcal{L}_{\mathrm{F}}| \\ |\mathcal{R}_{\mathrm{J}}(00)\rangle = |\mathcal{R}_{\mathrm{J}}(10)\rangle = |\mathcal{R}_{\mathrm{J}}(01)\rangle = |\mathcal{R}_{\mathrm{J}}(11)\rangle = |\mathcal{R}_{\mathrm{J}}\rangle \end{aligned} \tag{32}$$

and that the bulk matrices must satisfy

$$\sum_{p=0}^{n-1} \alpha^p \beta^{2(n-p-1)} F^{3p+2} \left(\beta^2 \mathcal{S}_3 J^2 + \beta F \mathcal{S}_2 J + F^2 \mathcal{S}_1\right) J^{3(n-p)-1} + \alpha^n F^{3n+2} \mathcal{S}_3 J$$

$$= \sum_{p=0}^{n} \alpha^p \beta^{2(n-p)} F^{3p} \left(\beta^2 \mathcal{S}_3 J^2 + \beta F \mathcal{S}_2 J + F^2 \mathcal{S}_1\right) J^{3(n-p)+1} + \alpha^{n+1} F^{3(n+1)} \mathcal{S}_3$$

$$+ (1-\alpha)^2(1-\beta) \sum_{p=0}^{n} \alpha^p \beta^{2(n-p)} F^{3p+2} \mathcal{S}_3 J^{3(n-p)+1}$$

$$+ (1-\alpha)(1-\beta) \sum_{p=0}^{n} \alpha^p \beta^{2(n-p)} F^{3p+1} \left(\beta \mathcal{S}_3 J + F \mathcal{S}_2\right) J^{3(n-p)+1}. \tag{33}$$

The requirement that the four sums in (33) cancel term by term leads to the following equation:

$$F^2 \left(\beta^2 \mathcal{S}_3 J^2 + \beta F \mathcal{S}_2 J + F^2 \mathcal{S}_1\right) = \beta^2 \left(\beta^2 \mathcal{S}_3 J^2 + \beta F \mathcal{S}_2 J + F^2 \mathcal{S}_1\right) J^2$$

$$+ \beta^2(1-\alpha)(1-\beta) F \left((1-\alpha) F \mathcal{S}_3 + \beta \mathcal{S}_3 J + F \mathcal{S}_2\right) J^2. \tag{34}$$

For the remaining terms in (33) to cancel, the following equation must be satisfied:

$$F^2 \mathcal{S}_3 J = \alpha F^3 \mathcal{S}_3 + \left(\beta^2 \mathcal{S}_3 J^2 + \beta F \mathcal{S}_2 J + F^2 \mathcal{S}_1 \right) J$$
$$+ (1 - \alpha)(1 - \beta) F \left((1 - \alpha) F \mathcal{S}_3 + \beta \mathcal{S}_3 J + F \mathcal{S}_2 \right) J. \tag{35}$$

Altogether we get three relations, (31), (34) and (35). These can be rewritten as

$$\mathcal{S}_2 J = \alpha F (\mathcal{S}_2 + (1 - \alpha) \mathcal{S}_3)$$
$$0 = \beta (\alpha \beta \mathcal{S}_3 + \mathcal{S}_2) J + F \mathcal{S}_1 \tag{36}$$
$$\alpha F^2 \mathcal{S}_3 J = \alpha \beta^2 \mathcal{S}_3 J^3 + F ((1 - \alpha)(1 - \beta) \mathcal{S}_2 + \alpha \beta (1 - \alpha - \beta) \mathcal{S}_3) J^2 + \alpha^2 F^3 \mathcal{S}_3.$$

Besides these bulk relations, there are boundary relations that follow from considerations of cases where f and/or j is close to the boundary. They are not particularly illuminating and are listed in appendix A. It is important to note that if we manage to solve these boundary relations such that (32) is also satisfied, a solution of (36) then ensures stationarity of the matrix product state (28) for arbitrary system sizes. This also means that (32) and (36) enable us to extrapolate knowledge of small systems to arbitrary large ones, which helps us to find a solution of the relations we have obtained. To find a representation for the matrices F and J, we employ the usual strategy for these type of problems: to consider explicit solutions for small systems and to try to find relations between them. Using the Ansatz (29) for the particular form of the matrices, we then deduce that the following algebraic relations hold:

$$0 = F^3 - \beta (1 - \alpha - \alpha \beta) F^2 - \alpha \beta^2 (2 - \alpha) F - \alpha^2 \beta^4$$
$$= (F - \beta)(F + \alpha \beta)(F + \alpha \beta^2) - \alpha \beta^2 (1 - \alpha)(1 - \beta) F \tag{37}$$

while J satisfies

$$0 = J^3 - \alpha (2 - \alpha) J^2 + \alpha^2 \beta (1 - \alpha - \alpha \beta) J + \alpha^4 \beta^2$$
$$= (J - \alpha)(J - \alpha \beta)(J - \alpha^2 \beta) + \alpha (1 - \alpha)(1 - \beta) J^2. \tag{38}$$

When considered as polynomials each of these equations has three solutions. These may be thought of as being the eigenvalues of F and J, respectively. One thus finds a three-dimensional representation for F and J for which we have to check that it is compatible with the other relations. This is indeed the case and two examples of explicit representations for all objects are given in appendix B. For $\alpha, \beta \neq 0$, F will have non-zero eigenvalues and thus is invertible. It is then found that (38) is satisfied by choosing $J = -\alpha^2 \beta^2 F^{-1}$.

Although the solutions of the cubic equation (37) are awkward expressions in terms of α and β, they are all real for $0 \leqslant \alpha, \beta \leqslant 1$. Let λ_n denote the eigenvalues of F and $\mu_n = -\alpha^2 \beta^2 / \lambda_n$ the eigenvalues of J. They can be written in the following way:

$$\lambda_n = a + \rho \cos \left(\frac{\phi + 2\pi n}{3} \right) \tag{39}$$

$$\mu_n = b + \rho' \sin \left(\frac{\phi' - (2n + 1)\pi}{3} \right) \tag{40}$$

with $\lambda_1 < \lambda_2 < 0 < \lambda_3$ and $\mu_3 < 0 < \mu_1 < \mu_2$ for $0 < \alpha, \beta < 1$. Here, a, b, ρ, ρ', ϕ and ϕ' are real functions of α and β defined in appendix B.1.

4. Results

In this section some exact results concerning the phase diagram are calculated. In the case of the purely free flow and jammed flow phases, the current and density profiles can be calculated easily. For general values of the boundary rates the correlation length is calculated and it is shown that it diverges on a special line in the phase diagram.

4.1. Free and jammed flow

We have seen that the stationary state in both the free flow case ($\beta = 1$) and the jammed flow case ($\alpha = 1$) is a simple product. This means that correlations are absent and the density profile is flat in both cases. The values of the current and the density are easily calculated and given by

$$\rho_F = \frac{\alpha}{1 + 2\alpha} \qquad j_F = \frac{2\alpha}{1 + 2\alpha} \tag{41}$$

$$\rho_J = \frac{1 - \beta}{1 - \beta^3} \qquad j_J = \frac{\beta(1 - \beta^2)}{1 - \beta^3}. \tag{42}$$

It follows that in the free flow phase $j_F = 2\rho_F$. As expected all particles move with a maximum speed of $v_{max} = 2$. In the jammed flow phase, the fundamental diagram is given by $j_J = 1 - \rho_J$. In this case all holes move with their maximum speed which is equal to 1. Note that the values of the two currents are equal if $2\alpha = \beta(1 + \beta)$. We will see in the next section that this line is the coexistence line in the general phase diagram.

4.2. The general case

To facilitate summing over the stationary state probabilities, the local free and jammed flow configuration weights $p_F(\tau\tau'\tau'')$ and $p_F(\tau\tau'\tau'')$ are collected into matrices P_f and P_j with the elements

$$\begin{aligned}\left(P_{f,j}\right)_{\tau\tau',\tau'\tau''} &= p_{f,j}(\tau\tau'\tau'') \\ &= 0 \quad \text{otherwise.}\end{aligned} \tag{43}$$

Likewise, the matrix S and vectors $\langle L|$ and $|R\rangle$ are defined

$$S_{\tau\tau',\tau'\tau''} = \mathcal{S}(\tau\tau'\tau'') \qquad \langle L|_{\tau\tau'} = \langle \mathcal{L}(\tau\tau')| \qquad |R\rangle_{\tau\tau'} = |R(\tau\tau')\rangle. \tag{44}$$

The advantage of this notation is that, for example, the normalization Z_L can be written compactly as

$$Z_L = \sum_{\{\tau\}} P(\tau_1, \ldots, \tau_L) = \langle L|M^{L-2}|R\rangle \tag{45}$$

where M is a 24×24 matrix given by

$$M = \begin{pmatrix} P_f \otimes F & S \\ 0 & P_j \otimes J \end{pmatrix}. \tag{46}$$

To perform the calculations, it is worth mentioning the following two intermediate results:

$$\langle L_F| (P_f \otimes F)^n = \alpha\beta^2(1 - \alpha)(1 - \beta)(1, 1, 1, 0) \otimes \left(\frac{\lambda_1^{n+1}}{\lambda_1 - \beta}, \frac{\lambda_2^{n+1}}{\lambda_2 - \beta}, \frac{\lambda_3^{n+1}}{\lambda_3 - \beta} \right) \tag{47}$$

$$\left((P_j \otimes J)^n |R_J\rangle \right)^t = -\frac{\beta}{\alpha\Delta}(1, 1, 1, 1) \otimes \left(\mu_1^{n+2}, \mu_2^{n+2}, \mu_3^{n+2} \right) \tag{48}$$

where the discriminant Δ is defined by

$$\Delta = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1). \tag{49}$$

After some laborious manipulations, the normalization is then found to be

$$\begin{aligned} Z_L = \frac{\beta^2(1 - \alpha)(1 - \beta)}{\alpha(2\alpha - \beta(1 + \beta))\Delta} \Bigg[&\alpha^2\beta(1 - \alpha)(1 + 2\alpha) \sum_{i=1}^{3} \frac{\lambda_i + \beta^2}{\lambda_i - \beta}(\lambda_{i+1} - \lambda_{i+2})\lambda_i^L \\ &+ (1 - \beta^3) \sum_{i=1}^{3} \frac{(\mu_i - 2\alpha)(\mu_i - \alpha^2)}{(\mu_i - \alpha)^2}(\mu_{i+1} - \mu_{i+2})\mu_i^{L+1} \Bigg] \end{aligned} \tag{50}$$

where $\lambda_{3+i} = \lambda_i$ and $\mu_{i+3} = \mu_i$. This expression is well defined for all α and β. To see this, the only non-trivial case we have to consider is $2\alpha = \beta(1+\beta)$. All other explicit poles in (50) are equivalent to $(1-\alpha)(1-\beta) = 0$. At $2\alpha = \beta(1+\beta)$ the solutions λ_i simplify dramatically. In particular $\lambda_1 = -\beta^2$ while λ_2 and λ_3 are the roots of a quadratic equation. Furthermore, we find that $\lambda_3 = \mu_2$ and $\lambda_2 = \mu_3$. These values imply that the expression between brackets in (50) has a zero that precisely cancels the pole at $2\alpha = \beta(1+\beta)$.

Unfortunately the calculations using the present notation are still rather intricate and laborious and to date no other explicit expressions for the general case have been obtained. The phase behaviour of the model, however, is similar to that of the case $v_{\max} = 1$ [17, 18, 41]. From the explicit form of the normalization it is clear that the correlation length will be determined by ratio of its largest contributions and we find

$$\xi_{\mathrm{F}}^{-1} = \ln \frac{\lambda_3}{\mu_2} = \ln \frac{\lambda_1}{-\beta^2} \qquad \text{for} \quad 2\alpha < \beta(1+\beta) \tag{51}$$

$$\xi_{\mathrm{J}}^{-1} = -\xi_{\mathrm{F}}^{-1} \qquad \text{for} \quad 2\alpha > \beta(1+\beta). \tag{52}$$

There is a low-density phase for $2\alpha < \beta(1+\beta)$ and a high density phase for $2\alpha > \beta(1+\beta)$. In each case, the bulk density will have the free and jammed flow value, respectively. At the boundaries there will be exponential corrections of which the correlation lengths are given by (51) and (52). The curve $2\alpha = \beta(1+\beta)$ is a coexisting curve on which the correlation length diverges. The instantaneous density profile is a shock profile resulting in an average linear profile. Across this curve, the average bulk density has a jump of size $\rho_{\mathrm{J}} - \rho_{\mathrm{F}}$.

As we have seen above, the locus of the coexisting curve is obtained by equating the values of the current for the two extreme cases: the free and jammed flow phases. This seems to be a general feature of ASEPs and supports the ideas of Kolomeisky *et al* [46] for the case of discrete time and parallel update. In the present model, however, the dependence of the correlation length on α and β does not decouple, as is the case for $v_{\max} = 1$ [18] and the random sequential ASEP [8]. It is therefore quite amazing that the locus of the coexisting curve still can be obtained by a simple mean field analysis.

5. Conclusion

An exact solution for the stationary state of a deterministic traffic model with $v_{\max} = 2$ is presented. Apart from the absence of symmetry due to the lack of a particle–hole duality, the phase diagram is qualitatively similar to that of the case with $v_{\max} = 1$, as expected. This solution might be a first step towards an exact solution of a realistic traffic model.

The stationary state is presented in a matrix product form, where the matrices depend on three sites and are 24-dimensional. The matrix product method has been shown to work extremely well in those cases where the matrices are either infinite-dimensional or of small finite dimension. Its shortcomings are obvious when the matrices are of large finite dimension and the eigenvalues become solutions of polynomials of high degree. As in the present case, it will still be a tedious technical exercise to derive exact expressions for expectation values and correlation functions from the exact solution.

Although many relations between matrix elements have been given, no proper matrix algebra has been derived. It will be interesting to find this underlying algebra, which at least should be of degree 4 as suggested by equation (36). This algebra may provide more convenient ways of deriving expectation values and correlation functions than the method used in this paper. The appearance of cubic roots, however, will remain.

An obvious and interesting extension of the model will be to include stochasticity in the bulk hopping rates. This can be done as in the Fukui–Ishibashi model [44], but more interesting

perhaps will be the aggressive driver Nagel–Schreckenberg model [47], which is closer to real traffic but might be still simple enough to be analytically tractable.

As a final remark one should mention that because of the long-range interaction it is unlikely that these models are integrable in the sense that there would be an underlying Yang–Baxter relation. The paradigmatic ASEP, however, is integrable since it is closely related to the integrable *XXZ* spin chain. It would be interesting to compare the matrix product ground states of non-integrable systems with those of integrable ones [37, 48].

Acknowledgments

Appendix A. Boundary relations

In the cases where $f \leqslant 3$ extra boundary relations are needed for (28) to be the stationary state. The following equations can be deduced from the master equation in those cases

- $f = 3$:

$$\alpha \langle \mathcal{L}_{\mathrm{F}} | F \mathcal{S}_3 J = \alpha \beta^2 \langle \mathcal{L}_{\mathrm{J}}(01) | J^3 + (1-\alpha)(1-\beta) \langle \mathcal{L}_{\mathrm{F}} | \mathcal{S}_2 J^2$$
$$+ \alpha \beta (1-\alpha-\beta) \langle \mathcal{L}_{\mathrm{F}} | \mathcal{S}_3 J^2 + \alpha^2 \langle \mathcal{L}_{\mathrm{F}} | F^2 \mathcal{S}_3. \tag{53}$$

- $f = 2$:

$$\alpha \beta (1-\beta) \langle \mathcal{L}_{\mathrm{J}}(01) | J = \beta \langle \mathcal{L}_{\mathrm{J}}(10) | J + \langle \mathcal{L}_{\mathrm{F}} | \mathcal{S}_1. \tag{54}$$

$$\alpha \langle \mathcal{L}_{\mathrm{F}} | \mathcal{S}_3 J = \alpha \beta^2 \langle \mathcal{L}_{\mathrm{J}}(01) | J^3 + \alpha (1-\alpha)(1-\beta)(\langle \mathcal{L}_{\mathrm{F}} | \mathcal{S}_2 + (1-\alpha) \langle \mathcal{L}_{\mathrm{F}} | \mathcal{S}_3) J$$
$$+ \alpha \beta (1-\beta) \langle \mathcal{L}_{\mathrm{J}}(01) | J^2 + \alpha^2 \langle \mathcal{L}_{\mathrm{F}} | F \mathcal{S}_3. \tag{55}$$

- $f = 1$:

$$\beta^2 \langle \mathcal{L}_{\mathrm{J}}(01) | J^2 + \langle \mathcal{L}_{\mathrm{F}} | (\beta \mathcal{S}_2 J + F \mathcal{S}_1) = \beta^2 (1-\beta) \left((1-\alpha) \langle \mathcal{L}_{\mathrm{J}}(01) | + \langle \mathcal{L}_{\mathrm{J}}(10) | \right) J^2 \tag{56}$$

$$\langle \mathcal{L}_{\mathrm{J}}(01) | J = \alpha \langle \mathcal{L}_{\mathrm{F}} | \mathcal{S}_3 + (1-\beta) \left((1-\alpha) \langle \mathcal{L}_{\mathrm{J}}(01) + \langle \mathcal{L}_{\mathrm{J}}(10) | \right) J. \tag{57}$$

- $f = 0$:

$$\langle \mathcal{L}_{\mathrm{J}}(11) | = \frac{\alpha}{\beta} (1-\beta) \langle \mathcal{L}_{\mathrm{J}}(01) | \qquad \langle \mathcal{L}_{\mathrm{F}}(11) | = 0 \tag{58}$$

$$\beta^2 \langle \mathcal{L}_{\mathrm{J}}(11) | J^2 = \beta \langle \mathcal{L}_{\mathrm{J}}(10) | J^2 + \langle \mathcal{L}_{\mathrm{F}} | \mathcal{S}_1 J. \tag{59}$$

Similarly, when $j \geqslant L - 1$ extra relations are needed. These are

- $j = L - 1$:

$$\mathcal{S}_2 | \mathcal{R}_{\mathrm{J}} \rangle = \alpha (1-\alpha) | \mathcal{R}_{\mathrm{F}}(00) \rangle. \tag{60}$$

- $j = L$:

$$F^2 \mathcal{S}_3 | \mathcal{R}_{\mathrm{J}} \rangle = \alpha F^3 | \mathcal{R}_{\mathrm{F}}(00) \rangle + (\beta^2 \mathcal{S}_3 J^2 + \beta F \mathcal{S}_2 J + F^2 \mathcal{S}_1) | \mathcal{R}_{\mathrm{J}} \rangle$$
$$+ (1-\alpha)(1-\beta) F ((1-\alpha) F \mathcal{S}_3 + \beta \mathcal{S}_3 J + F \mathcal{S}_2) | \mathcal{R}_{\mathrm{J}} \rangle. \tag{61}$$

- $j = L + 1$:

$$(1 - (1-\alpha)^2) F^2 | \mathcal{R}_{\mathrm{F}}(00) \rangle = \beta (\beta \mathcal{S}_3 J + F \mathcal{S}_2 + (1-\alpha) F \mathcal{S}_3) | \mathcal{R}_{\mathrm{J}} \rangle. \tag{62}$$

Appendix B. Representations

B.1. The diagonal representation

In this subsection an explicit representation is given in which F and J are diagonal. Let λ_1, λ_2 and λ_3 denote the three solutions of the equation

$$\lambda^3 - 3a\lambda^2 - 3b\lambda - \alpha\beta^2 c = 0 \tag{63}$$

where

$$
\begin{aligned}
a &= (\lambda_1 + \lambda_2 + \lambda_3)/3 = \beta(1 - \alpha - \alpha\beta)/3 \\
b &= -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)/(3\beta^2) = \alpha(2 - \alpha)/3 \\
c &= \lambda_1\lambda_2\lambda_3/\beta^2 = \alpha^2\beta^2.
\end{aligned}
\tag{64}
$$

Then, F and J are given by

$$F = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\} \qquad J = -\alpha^2\beta^2\mathrm{diag}\{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\}. \tag{65}$$

The matrices S_i are given by

$$
\begin{aligned}
S_1 &= \alpha^2\beta^2 \begin{pmatrix} 0 & -\lambda_1 - \lambda_2 & \lambda_1 + \lambda_3 \\ \lambda_1 + \lambda_2 & 0 & -\lambda_2 - \lambda_3 \\ -\lambda_1 - \lambda_3 & \lambda_2 + \lambda_3 & 0 \end{pmatrix} \\
S_2 &= (1 - \alpha) \begin{pmatrix} 0 & -\lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & 0 & -\lambda_2\lambda_3 \\ -\lambda_1\lambda_3 & \lambda_2\lambda_3 & 0 \end{pmatrix} \\
S_3 &= \begin{pmatrix} 0 & \lambda_1\lambda_2 + \alpha\beta^2 & -\lambda_1\lambda_3 - \alpha\beta^2 \\ -\lambda_1\lambda_2 - \alpha\beta^2 & 0 & \lambda_2\lambda_3 + \alpha\beta^2 \\ \lambda_1\lambda_3 + \alpha\beta^2 & -\lambda_2\lambda_3 - \alpha\beta^2 & 0 \end{pmatrix}.
\end{aligned}
\tag{66}
$$

The corresponding boundary vectors are given by

$$
\begin{aligned}
\langle \mathcal{L}_F| &= \alpha\beta^2(1 - \alpha)(1 - \beta)\left(\frac{\lambda_1}{\lambda_1 - \beta}, \frac{\lambda_2}{\lambda_2 - \beta}, \frac{\lambda_3}{\lambda_3 - \beta}\right) \\
\langle \mathcal{L}_F(11)| &= 0 \\
\langle \mathcal{L}_J(10)| &= \big((\lambda_1 - \beta)(\lambda_2 - \lambda_3)((2\alpha - 1)\lambda_1 + \alpha^2\beta), (\lambda_2 - \beta)(\lambda_3 - \lambda_1) \\
&\quad ((2\alpha - 1)\lambda_2 + \alpha^2\beta), (\lambda_3 - \beta)(\lambda_1 - \lambda_2)((2\alpha - 1)\lambda_3 + \alpha^2\beta)\big) \\
\langle \mathcal{L}_J(01)| &= \alpha\beta^2(1 - \alpha)(1 - \beta)\left(\frac{\lambda_1(\lambda_2 - \lambda_3)}{\lambda_1 + \alpha\beta^2}, \frac{\lambda_2(\lambda_3 - \lambda_1)}{\lambda_2 + \alpha\beta^2}, \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_3 + \alpha\beta^2}\right) \\
\langle \mathcal{L}_J(11)| &= \frac{\alpha}{\beta}(1 - \beta)\langle L_J(01)|
\end{aligned}
\tag{67}
$$

and the right boundary vectors are given by

$$
\begin{aligned}
|\mathcal{R}_J\rangle &= -\frac{\alpha^3\beta^5}{\Delta}\left(\lambda_1^{-2}, \lambda_2^{-2}, \lambda_3^{-2}\right) \\
|\mathcal{R}_F(00)\rangle &= -\frac{\beta}{\Delta}\left(\lambda_1(\lambda_2 - \lambda_3), \lambda_2(\lambda_3 - \lambda_1), \lambda_3(\lambda_1 - \lambda_2)\right) \\
|\mathcal{R}_F(10)\rangle &= |R_F(01)\rangle = |R_F(11)\rangle = 0
\end{aligned}
\tag{68}
$$

where the discriminant Δ is defined in (49).

The solutions of (63) are all real for $0 < \alpha, \beta < 1$ and can be written in the following form:

$$\lambda_n = a + \rho\cos\left(\frac{\phi + 2\pi n}{3}\right) \tag{69}$$

where

$$\rho = 2\sqrt{a^2 + \beta^2 b} \qquad \phi = \arctan\left(\Delta / C\right). \tag{70}$$

The discriminant Δ and C are given by

$$\Delta = \sqrt{27\rho^6/16 - C^2} \qquad C = 3\sqrt{3}(2a^3 + \beta^2(c + 3ab)). \tag{71}$$

Similarly, the solutions $\mu_n = -\alpha^2\beta^2/\lambda_n$ of (38) can be written as

$$\mu_n = b + \rho' \sin\left(\frac{\phi' - (2n + 1)\pi}{3}\right) \tag{72}$$

where

$$\rho' = 2\sqrt{b^2 - \alpha^2 a} \qquad \phi' = \arctan\left(C'/\Delta\right). \tag{73}$$

$$C' = 3\sqrt{3}\beta^2(2b^3 - \alpha^2(c + 3ab))/\alpha^2. \tag{74}$$

B.2. A simple representation

Although the eigenvalues of F and J, or equivalently the roots of (37) and (38), are awkward expressions, an example of a representation with simple matrix elements is given by

$$F = \beta \begin{pmatrix} 1 & 0 & \alpha \\ 1 - \alpha & -\alpha\beta & \alpha(1 - \alpha) \\ 0 & 1 - \beta & -\alpha \end{pmatrix} \qquad J = \alpha \begin{pmatrix} 0 & \alpha\beta & 0 \\ \beta & 1 - \alpha & 1 - \beta \\ 0 & 1 - \alpha & 1 \end{pmatrix}. \tag{75}$$

The corresponding representations for $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are given by

$$\mathcal{S}_1 = -\frac{\alpha\beta^2}{1 - \beta} \begin{pmatrix} \beta(1 - \alpha) & 1 - \alpha & 1 - \beta \\ \alpha\beta(1 - \beta) & \alpha(1 - \alpha)(1 - \beta) & \alpha(1 - \beta)^2 \\ \alpha\beta(1 - \alpha) & \alpha(1 - \alpha)^2 & \alpha(1 - \alpha)(1 - \beta) \end{pmatrix}$$

$$\mathcal{S}_2 = \frac{\beta^2}{1 - \beta} \begin{pmatrix} 1 - \alpha & 1 - \alpha & 0 \\ 0 & \alpha(1 - \alpha)(1 - \beta) & 0 \\ \alpha(1 - \alpha) & \alpha(1 - \alpha)^2 & 0 \end{pmatrix} \qquad \mathcal{S}_3 = \beta \begin{pmatrix} 0 & 1 - \alpha & 1 \\ 0 & -\alpha\beta & 0 \\ 0 & \alpha(1 - \alpha) & \alpha \end{pmatrix}. \tag{76}$$

The left boundary vectors in this representation are represented by

$$\langle \mathcal{L}_F| = (1, 1 - \alpha, 0)$$
$$\langle \mathcal{L}_J(10)| = \frac{1}{1 - \beta}(\beta(1 - \alpha), (1 - \alpha)(\alpha + \beta - \alpha\beta), \alpha(1 - \beta))$$
$$\langle \mathcal{L}_J(00) = 0 \qquad \langle \mathcal{L}_J(01)| = (0, 1 - \alpha, 1) \tag{77}$$
$$\langle \mathcal{L}_J(11)| = \frac{\alpha}{\beta}(1 - \beta)\langle \mathcal{L}_J(01)|$$

and the right boundary vectors are given by

$$|\mathcal{R}_J\rangle = (0, 1 - \beta, 1)$$
$$|\mathcal{R}_F(10)\rangle = |\mathcal{R}_F(01)\rangle = |\mathcal{R}_F(11)\rangle = 0 \qquad |\mathcal{R}_F(00)\rangle = \frac{\beta}{\alpha}(1, 0, \alpha). \tag{78}$$

This representation is convenient for calculations for small system sizes. For larger system sizes it is more useful to use a representation in which F and J are diagonal. The price one has to pay is that the matrix elements will be more complicated because they are cubic roots.

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